第一章 方程的定解问题和导出

将方程化为标准型：

$$a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y + f u = g$$

方程的特征方程：$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}$

① $\Delta = b^2 - ac$ $\Rightarrow$ 两根　双曲型

$\xi(x,y) = C_1$，$\eta(x,y) = C_2$

原方程 $\Rightarrow u_{\xi\eta} = H_1(\xi, \eta, u, u_\xi, u_\eta)$

$$\Rightarrow \begin{cases} \xi = s + t \\ \eta = s - t \end{cases} \quad \left(\begin{matrix} s = \xi + \eta \\ t = \xi - \eta \end{matrix} \right)$$

$$\Rightarrow u_{ss} - u_{tt} = H_2(s, t, u, u_s, u_t)$$

例：$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = \frac{4}{9}$

$\Delta = \left(\frac{5}{2}\right)^2 - 4 > 0 \Rightarrow$ 双曲型

$$\frac{dy}{dx} = \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4}}{4}$$

$$\begin{cases} \frac{dy}{dx} = 1 \Rightarrow y - x = C_1 \\ \frac{dy}{dx} = \frac{1}{4} \Rightarrow y - \frac{1}{4}x = C_2 \end{cases}$$

令 $\begin{cases} \xi = y - x \\ \eta = y - \frac{1}{4}x \end{cases}$ $\xRightarrow[\text{化为}]{\text{方程}}$ $u_{\xi\eta} = \frac{1}{3}u_\eta + \frac{4}{3}$

作 $\begin{cases} \xi = s + t \\ \eta = s - t \end{cases}$ 再化 $u_{ss} - u_{tt} = \frac{2}{3}(u_s - u_t) - 4$

(2) $\Delta = 0$　只有一个实根　抛物型

$\xi(x,y) = C_1$，取与其线性无关的任一函数 $\eta = \eta(x,y)$

方程可化简为 $u_{\eta\eta} = H_3(\xi, \eta, u, u_\xi, u_\eta)$

例：$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = 0$

$\Delta = x^2y^2 - x^2y^2 = 0$

$$\frac{dy}{dx} = \frac{xy \pm 0}{x^2} = \frac{y}{x}$$

可分离变量 $\frac{1}{y}dy = \frac{1}{x}dx$

$\Rightarrow \ln y = \ln x + c$

$\therefore \frac{y}{x} = C_1$ (特征线)

令 $\begin{cases} \xi = \frac{y}{x} \\ \eta = y \end{cases}$ $\Rightarrow y^2 u_{\eta\eta} = 0$

即 $u_{\eta\eta} = 0$

(3) $\Delta<0$ 这时方程有一对共轭复根 椭圆型

解得通积分 $\xi(x,y)=C_1$, $\eta(x,y)=C_2$, 其中 $\eta=\bar{\xi}$

引进替换 $\begin{cases}\xi=s+ir\\ \eta=s-ir\end{cases} \Rightarrow u_{ss}+u_{rr}=H_4(s,r,u,u_s,u_r)$

例: $yU_{xx}+U_{yy}=0 \quad (y>0)$

$$\frac{dy}{dx}=\frac{\pm\sqrt{-y}}{y}=\frac{\pm i}{\sqrt{y}}$$

$$\therefore \begin{cases}\frac{2}{3}y^{\frac{3}{2}}-ix=C_1\\ \frac{2}{3}y^{\frac{3}{2}}+ix=C_2\end{cases} \Rightarrow \text{取} \begin{cases}\xi=x\\ \eta=\frac{2}{3}y^{\frac{3}{2}}\end{cases}$$

$$\Rightarrow U_{\xi\xi}+U_{\eta\eta}+\frac{1}{3\eta}U_\eta=0$$

第二章 行波法

一、无界弦的自由振动

$$\begin{cases} u_{tt} = a^2 u_{xx}, & (-\infty < x + \infty, t > 0) \\ u|_{t=0} = \varphi(x), & (-\infty < x + \infty) \\ u_t|_{t=0} = \psi(x) & (-\infty < x < +\infty) \end{cases}$$

特征方程：$\left(\frac{dx}{dt}\right)^2 - a^2 = 0$

特征线：$\begin{cases} x - at = c_1 \\ x + at = c_2 \end{cases} \overset{令}{\Longrightarrow} \begin{cases} \xi = x - at \\ \eta = x + at \end{cases} \overset{原方程}{\Longrightarrow} u_{\xi\eta} = 0$

关于 η 积一次分 $\frac{\partial u}{\partial \xi} = c(\xi)$

关于 ξ 积一次分 $u(\xi, \eta) = F(\xi) + G(\eta) \underset{旧变量}{\overset{换回}{\Longrightarrow}} u(x,t) = F(x-at) + G(x+at)$

利用初值条件可解得 $F(x), G(x)$

结论：达朗贝尔公式

$$u(x,t) = \frac{1}{2}\left(\varphi(x-at) + \varphi(x+at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\alpha)\, d\alpha$$

例 $\begin{cases} u_{tt} = a^2 u_{xx} & (-\infty < x + \infty, t > 0) \\ u|_{t=0} = \cos x & (-\infty < x + \infty) \\ u_t|_{t=0} = Ax & (-\infty < x + \infty) \end{cases}$

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\varphi(x+at) + \varphi(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\alpha)\, d\alpha \\ &= \frac{1}{2}\left(\cos(x+at) + \cos(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} A\alpha\, d\alpha \\ &= \cos x \cos at + Axt \end{aligned}$$

二、半无界弦的自由振动

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} & (0<x<+\infty,\ t>0) \\ u|_{t=0}=\varphi(x) & (0<x<+\infty) \\ \dfrac{\partial u}{\partial t}\Big|_{t=0}=\psi(x) & (0<x<+\infty) \\ u|_{x=0}=0 & (t\geq 0) \end{cases}$$

$u_x|_{x=0}=0 \quad (t\geq 0)$

① 奇延拓

$$U(x,t)=\begin{cases} u(x,t) & (x\geq 0) \\ -u(-x,t) & (x<0) \end{cases}$$

$$\Phi(x,t)=\begin{cases} \varphi(x,t) & (x\geq 0) \\ -\varphi(-x,t) & (x<0) \end{cases}$$

$$\Psi(x,t)=\begin{cases} \psi(x,t) & (x\geq 0) \\ -\psi(-x,t) & (x<0) \end{cases}$$

大写 小写

利用达朗贝尔公式

$$U(x,t)=\frac{1}{2}\left(\Phi(x+at)+\Phi(x-at)\right)+\frac{1}{2a}\int_{x-at}^{x+at}\Psi(\alpha)\,d\alpha$$

考虑 $x\geq 0$ 时

(i) 若 $x-at\geq 0$

$$u(x,t)=\frac{1}{2}\left(\varphi(x+at)+\varphi(x-at)\right)+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)\,d\alpha$$

(ii) 若 $x-at<0$

大写 小写

$$\int_{x-at}^{x+at}\Psi(\alpha)\,d\alpha=\int_{x-at}^{0}\Psi(\alpha)\,d\alpha+\int_{0}^{x+at}\psi(\alpha)\,d\alpha$$

这里是取 $x<0$ 时的函数

小写

$$=\int_{x-at}^{0}\left(-\psi(-\alpha)\right)d\alpha+\int_{0}^{x+at}\psi(\alpha)\,d\alpha$$

变量替换

$$\overset{\text{令}\alpha=-\alpha}{=}\int_{at-x}^{0}\psi(\alpha)\,d\alpha+\int_{0}^{x+at}\psi(\alpha)\,d\alpha$$

$$u(x,t)=\frac{1}{2}\left(\varphi(x+at)-\varphi(at-x)\right)+\frac{1}{2a}\int_{at-x}^{x+at}\psi(\alpha)\,d\alpha$$

② 偶延拓

$$U(x,t)=\begin{cases} u(x,t) & (x\geq 0) \\ u(-x,t) & (x<0) \end{cases}$$

$$\Phi(x,t)=\begin{cases} \varphi(x,t) & (x\geq 0) \\ \varphi(-x,t) & (x<0) \end{cases}$$

$$\Psi(x,t)=\begin{cases} \psi(x,t) & (x\geq 0) \\ \psi(-x,t) & (x<0) \end{cases}$$

利用达朗贝尔公式

$$U(x,t)=\frac{1}{2}\left(\Phi(x+at)+\Phi(x-at)\right)+\frac{1}{2a}\int_{x-at}^{x+at}\Psi(\alpha)\,d\alpha$$

(i) $x-at\geq 0$

$$u(x,t)=\frac{1}{2}\left(\varphi(x+at)+\varphi(x-at)\right)+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)\,d\alpha$$

(ii) $x-at<0$

$$\int_{x-at}^{x+at}\Psi(\alpha)\,d\alpha=\int_{x-at}^{0}\Psi(\alpha)\,d\alpha+\int_{0}^{x+at}\psi(\alpha)\,d\alpha$$

$$=\int_{x-at}^{0}\psi(-\alpha)\,d\alpha+\int_{0}^{x+at}\psi(\alpha)\,d\alpha$$

$$\overset{\text{令}\alpha=-\alpha}{=}\int_{at-x}^{0}\psi(\alpha)\,d(-\alpha)+\int_{0}^{x+at}\psi(\alpha)\,d\alpha$$

$$=\int_{0}^{at-x}\psi(\alpha)\,d\alpha+\int_{0}^{x+at}\psi(\alpha)\,d\alpha$$

$$\therefore u(x,t)=\frac{1}{2}\left(\varphi(x+at)+\varphi(at-x)\right)+\frac{1}{2a}\left(\int_{0}^{at-x}\psi(\alpha)\,d\alpha+\int_{0}^{x+at}\psi(\alpha)\,d\alpha\right)$$

三、无界弦的强迫振动

$$\begin{cases} u_{tt}=a^2u_{xx}+f(x,t) & (-\infty<x+\infty,\ t>0) \\ u|_{t=0}=\varphi(x) & (-\infty<x+\infty) \\ u_t|_{t=0}=\psi(x) & (-\infty<x+\infty) \end{cases}$$

利用线性叠加原理 $u=u_1+u_2$

u_1 为 $\begin{cases} u_{tt}=a^2u_{xx} & (-\infty<x<+\infty,\ t>0) \\ u|_{t=0}=\varphi(x) & (-\infty<x+\infty) \\ u_t|_{t=0}=\psi(x) & (-\infty<x+\infty) \end{cases}$ 的解

u_1 利用达朗贝尔公式可直接求解

u_2 为 $\begin{cases} u_{tt}=a^2u_{xx}+f(x,t) \\ u|_{t=0}=0 \\ u_t|_{t=0}=0 \end{cases}$ 的解

齐次化原理：设 $w(x,t,\tau)$ 是初值问题

$\begin{cases} w_{tt}=a^2U_{xx} & (-\infty<x+\infty,\ t>\tau) \\ w|_{t=\tau}=0 & (-\infty<x+\infty) \\ w_t|_{t=\tau}=f(x,\tau) & (-\infty<x+\infty) \end{cases}$ 的解

则 $u_2=\int_0^t w(x,t,\tau)\,d\tau$

解：令 $t'=t-\tau$ $\begin{cases} w_{t't'}=a^2w_{xx} & (-\infty<x+\infty,\ t'>0) \\ w|_{t'=0}=0 & (-\infty<x+\infty) \\ w_{t'}|_{t'=0}=f(x,\tau) & (-\infty<x<+\infty) \end{cases}$

$\therefore w(x,t,\tau)=\frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)\,d\xi$

$\therefore u_2=\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)\,d\xi\,d\tau$

综上 $u(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t+\tau)}\psi(\xi)\,d\xi$

例：利用齐次化原理求解

$$\begin{cases} u_{tt}=u_{xx}+t\sin x & (-\infty<x+\infty,\ t>0) \\ u|_{t=0}=0 & (-\infty<x<+\infty) \\ u_t|_{t=0}=0 & (-\infty<x+\infty) \end{cases}$$

令 $u=u_1+u_2$

求解 $u_1=0$

求解 u_2，应先求

$$\begin{cases} W_{tt}=W_{xx} & (-\infty<x<+\infty,\ t>\tau) \\ W|_{t=\tau}=0 & (-\infty<x+\infty) \\ W_t|_{t=\tau}=\tau\sin x & (-\infty<x+\infty) \end{cases}$$

令 $t'=t-\tau \Rightarrow \begin{cases} W_{t't'}=W_{xx} \\ W|_{t'=0}=0 \\ W_{t'}|_{t'=0}=\tau\sin x \end{cases}$

$$W=\frac{1}{2}\int_{x-(t+\tau)}^{x+(t-\tau)}\tau\sin\xi\, d\xi$$

$$=\tau\sin x\sin(t-\tau)$$

$$\therefore u=\int_0^t w\, d\tau$$

$$=\sin x\int_0^t \tau\sin(t-\tau)\, d\tau$$

$$=\sin x\,(t-\sin t)$$

第三章 分离变量法和特殊函数法

一、齐次方程齐次边界条件

(1) 有界弦的自由振动

$$\begin{cases} u_{tt}=a^2u_{xx}, & (0<x<l,\ t>0) \\ u|_{x=0}=0,\ u|_{x=l}=0 & \Leftarrow \text{第一类边界条件} \\ u|_{t=0}=\varphi(x),\ u_t|_{t=0}=\psi(x) & \Leftarrow \text{初值条件} \end{cases}$$

令 $u(x,t)=X(x)T(t)$

$$X(x)T''(t)=a^2X''(x)T(t)$$

$$\frac{T''(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}=-\lambda$$

求解本征值

由边界条件得到 ⇒ $\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=X(l)=0 \end{cases}$ 常微分方程知识：线性二阶常微分方程 ⇒

只有 $\lambda>0$ 时才成立 $\quad \Delta=\frac{\pm\sqrt{-4\lambda}}{2}=\pm\sqrt{\lambda}\,i$

$$\therefore X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x$$

代入边界条件 ⇒ $C_1=0,\ C_2\sin\sqrt{\lambda}l=0$

$\therefore C_2\neq 0,\ \sin\sqrt{\lambda}l=0 \quad \sqrt{\lambda}l=n\pi \Rightarrow \begin{cases} \lambda_n=\left(\frac{n\pi}{l}\right)^2 \\ X_n(x)=C_n\sin\frac{n\pi}{l}x \quad (n=1,2,\cdots) \end{cases}$

$$T''(t)+a^2\lambda_nT(t)=0$$

$$\Rightarrow T_n(t)=a_n\cos a\sqrt{\lambda_n}\,t+b_n\sin a\sqrt{\lambda_n}\,t$$

$$\therefore u_n(x,t)=X_n(x)T_n(t)$$

$$=\left(A_n\cos\frac{n\pi}{l}at+B_n\cos\frac{n\pi}{l}at\right)\sin\frac{n\pi}{l}x,\quad (n=1,2,\cdots)$$

叠加：$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)$

$$=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi}{l}at+B_n\sin\frac{n\pi}{l}at\right)\sin\frac{n\pi}{l}x$$

由初值条件：

$$\begin{cases}\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{l}x=\varphi(x)\\ \sum_{n=1}^{\infty}\frac{n\pi}{l}aB_n\sin\frac{n\pi}{l}x=\psi(x)\end{cases}\xrightarrow[\text{展开式}]{\text{傅里叶}}\begin{cases}A_n=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{n\pi}{l}x\,dx\\ B_n=\frac{2}{n\pi a}\int_0^l\psi(x)\sin\frac{n\pi}{l}x\,dx\end{cases}$$

$$\begin{cases}\text{第一类边界条件} & \lambda_n=\frac{n^2\pi^2}{l^2} & X_n(x)=C_n\sin\frac{n\pi}{l}x & (n=1,2,\cdots)\\ \text{第二类边界条件} & \lambda_n=\frac{n^2\pi^2}{l^2} & X_n(x)=C_n\cos\frac{n\pi}{l}x, & (n=0,1,2\cdots)\\ \text{第三类边界条件} & \lambda_n=\frac{(n-\frac{1}{2})^2\pi^2}{l^2} & X_n(x)=C_n\sin\frac{(n-\frac{1}{2})\pi}{l}x, & (n=1,2,\cdots)\end{cases}$$

例：
$$\begin{cases}u_t=a^2u_{xx}\\ u|_{x=0}=0,\ u_x|_{x=l}=0\\ u|_{t=0}=\varphi(x)\end{cases}$$
← 第三类边界条件

解 $U(x,t)=X(x)T(t)$

$T'(t)X(x)=a^2X''(x)T(t)$

$$\frac{T'(t)}{aT(t)}=\frac{X''(x)}{X(x)}=-\lambda$$

$$\begin{cases}X''(x)+\lambda X(0)=0\\ X(0)=X'(l)=0\end{cases}\Rightarrow\lambda_n=\frac{(n-\frac{1}{2})^2\pi^2}{l^2}$$

$\therefore X_n=C_n\sin\frac{(n-\frac{1}{2})\pi}{l}x\quad(n=1,2\cdots)$

$T'(t)=-\lambda aT(t)$

$\frac{1}{T(t)}dT(t)=-\lambda a\,dt$

$\ln T(t)=-\lambda at$

$\therefore T(t)=e^{-\frac{(n-\frac{1}{2})^2\pi^2}{l^2}at}$

$\therefore u(x,t)=a_n\sin\frac{(n-\frac{1}{2})\pi}{l}x\,e^{-\frac{(n-\frac{1}{2})^2\pi^2}{l^2}at}$

$\therefore U(x,t)=\sum_{n=1}^{\infty}A_ne^{-\lambda_na^2t}\sin\sqrt{\lambda_n}x$

初值代入：$\sum_{n=1}^{\infty}A_n\sin\frac{n-\frac{1}{2}}{l}\pi x=\varphi(x)$

$\therefore A_n=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{n-\frac{1}{2}}{l}\pi x\,dx$

二、非齐次方程齐次边界条件

$$\begin{cases} V_{tt}=a^2V_{xx}+f(x,t) \quad (0<x<l,\ t>0) \\ V|_{x=0}=0,\ V|_{x=l}=0 \\ V|_{t=0}=0,\ V_t|_{t=0}=0 \end{cases}$$

边界值得到本征函数系

方法一：本征纸展开法

设 $V(x,t)=\sum_{n=1}^{\infty}C_n(t)\sin\frac{n\pi}{l}x$，$f(x,t)=\sum_{n=1}^{\infty}f_n(t)\sin\frac{n\pi}{l}x$

其中，$f_n(t)=\frac{2}{l}\int_0^l f(x,t)\sin\frac{n\pi}{l}x\,dx$

原方程可化为 $\sum_{n=1}^{\infty}C_n''(t)\sin\frac{n\pi}{l}x=a^2\sum_{n=1}^{\infty}C_n(t)\frac{n^2\pi^2}{l^2}\times(-1)\times\sin\frac{n\pi}{l}x+\sum_{n=1}^{\infty}f_n(t)\sin\frac{n\pi}{l}x$

从初值求得

$$\therefore \begin{cases} C_n''(t)+\frac{n^2\pi^2a^2}{l}C_n(t)=f_n(t) \\ C_n(0)=C_n'(0)=0 \end{cases}$$

齐次化原理

$$\begin{cases} W_{tt}+\frac{n^2\pi^2a^2}{l^2}W_t=0 \\ W|_{t=\tau}=0 \\ W_t|_{t=\tau}=f_n(\tau) \end{cases} \xRightarrow{t'=t-\tau} \begin{cases} W_{t't'}+\frac{n^2\pi^2a^2}{l^2}W_{t'}=0 \\ W|_{t'=0}=0 \\ W_{t'}|_{t'=0}=f_n(\tau) \end{cases}$$

$\therefore W_n=C_1\cos\frac{n\pi a}{l}t'+C_2\sin\frac{n\pi a}{l}t'$，$t'=t-\tau$

$$\begin{cases} W|_{t'=0}=C_1=0 \\ W_{t'}|_{t'=0}=C_2\frac{n\pi a}{l}\cos\frac{n\pi a}{l}t'=\frac{n\pi a}{l}\cdot C_2=f_n(\tau) \Rightarrow C_2=f_n(\tau)\frac{l}{n\pi a} \end{cases}$$

$W_n(t,\tau)=\frac{l}{n\pi a}f_n(\tau)\sin\frac{n\pi a}{l}(t-\tau)$

$\therefore C_n(t)=\int_0^t\frac{l}{n\pi a}f_n(\tau)\sin\frac{n\pi a}{l}(t-\tau)\,dt$

代回原式即可解得 $V(x,t)=\sum_{n=1}^{\infty}C_n(t)\sin\frac{n\pi}{l}x$

方法二，直接利用齐次化原理

注：齐次化原理非常适合用来求解常系数非齐次方程，齐次边界条件，零初值的情况

解：先求 $W(x,t,\tau)$

$$\begin{cases} W_{tt}=a^2W_{xx} & (0<x<l,\ t>\tau) \\ W|_{x=0}=W|_{x=l}=0 \\ W|_{t=\tau}=0,\ W_t|_{t=\tau}=f(x,\tau) \end{cases}$$

令 $t'=t-\tau$ $\Rightarrow$
$$\begin{cases} W_{t't'}=a^2W_{xx} \\ W|_{x=0}=W|_{x=l}=0 \\ W|_{t'=0}=0,\ W_t|_{t'=0}=f(x,\tau) \end{cases}$$

$\therefore$ 令 $W=T(t')X(x)$

通过求解本征值问题可解得

$$X(x)=C_n\sin\frac{n\pi}{l}x$$

$$\frac{T''(t')}{a^2T(t')}=-\frac{n^2\pi^2}{l^2}\Rightarrow T''(t')+a^2\frac{n^2\pi^2}{l^2}T(t')=0$$

$$\Rightarrow T(t')=a_n\cos\frac{n\pi a}{l}t'+b_n\sin\frac{n\pi a}{l}t' \qquad \frac{n\pi a}{l}B_n\sin\frac{n\pi x}{l}=$$

$$\therefore W(t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi a}{l}t'+B_n\sin\frac{n\pi a}{l}t'\right)\sin\frac{n\pi x}{l}$$

代入初值 $A_n=0$

$$B_n=\frac{2}{n\pi a}\int_0^l f(x,\tau)\sin\frac{n\pi x}{l}dx$$

$$\therefore V(x,t)=\sum_{n=1}^{\infty}\int_0^t B_n\sin\frac{n\pi a}{l}(t-\tau)\sin\frac{n\pi x}{l}\,d\tau$$

$$\begin{cases} u_{tt}=a^2u_{xx}+f(x,t), & (0<x<l,\ t>0) \\ u|_{x=0}=u|_{x=l}=0 \\ u|_{t=0}=\varphi(x),\ u_t|_{t=0}=\psi(x) \end{cases}$$

法一：将 $\varphi(x)$, $\psi(x)$ 同时按本征函数展开

法二：利用叠加原理

$$u_1:\begin{cases} u_{tt}=a^2u_{xx} \\ u|_{x=0}=u|_{x=l}=0 \\ u|_{t=0}=\varphi(x),\ u_t|_{t=0}=\psi(x) \end{cases} \qquad u_2:\begin{cases} u_{tt}=a^2u_{xx}+f(x,t) \\ u|_{x=0}=u|_{x=l}=0 \\ u|_{t=0}=0,\ u_t|_{t=0}=0 \end{cases}$$

齐次方程齐次边界条件的解法　　　用齐次化原理求解

例
$$\begin{cases} u_t=u_{xx}+t\sin(\frac{5}{2}\pi x), & (0<x<l,\ t>0) \\ u|_{x=0}=0,\ u_x|_{x=1}=0 \\ u|_{t=0}=3\sin(\frac{11\pi x}{2}) \end{cases}$$

解：设解 $u=V(x,t)+h(x,t)$

$$\begin{cases} V_t=V_{xx} \\ V|_{x=0}=0,\ V_x|_{x=1}=0 \\ V|_{t=0}=3\sin(\frac{11\pi x}{2}) \end{cases}$$

分离变量：$V=\sum_{n=1}^{\infty}A_n e^{-(n\pi-\frac{1}{2}\pi)^2t}\sin\frac{(n-\frac{1}{2})\pi}{1}x$

其中 $A_n=2\int_0^1 3\sin(\frac{11}{2}\pi x)\sin(n\pi-\frac{\pi}{2})x\,dx=\begin{cases}3, & n=6\\ 0, & n\neq 6\end{cases}$ (正交性)

$\therefore V=3e^{-(6\pi-\frac{1}{2}\pi)^2t}\sin(6\pi-\frac{\pi}{2})x$

$$\begin{cases} h_t = h_{xx} + t\sin(\frac{5}{2}\pi x) \\ h|_{x=0} = 0,\ h_x|_{x=1} = 0 \\ h|_{t=0} = 0 \end{cases}$$

齐次化原理 $\begin{cases} w_t = w_{xx} \\ w|_{x=0} = 0,\ w_x|_{x=1} = 0 \\ w|_{t=\tau} = \tau\sin(\frac{5\pi x}{2}) \end{cases} \Rightarrow w = \sum_{n=1}^{\infty} B_n e^{-(n\pi - \frac{1}{2}\pi)^2 t'} \sin(n\pi - \frac{\pi}{2})x,\ t' = t-\tau$

$$B_n = 2\int_0^1 \tau\sin(\frac{5}{2}\pi x)\sin(n\pi - \frac{\pi}{2})x\,dx = \begin{cases} \tau, & n=3 \\ 0, & n\neq 3 \end{cases}$$

$\therefore w = \tau e^{-(3\pi - \frac{1}{2}\pi)^2(t-\tau)} \sin(3\pi - \frac{\pi}{2})x$

$\therefore h(x,t) = \int_0^t w\,d\tau$ 最后两者相加即得 u

三、非齐次边界条件的定解问题

边界条件齐次化

$$\begin{cases} u_{tt}=a^2u_{xx}+f(x,t) & (0<x<l,\ t>0) \\ u|_{x=0}=\mu(t),\ u|_{x=l}=\nu(t) \\ u|_{t=0}=\varphi(x),\ u_t|_{t=0}=\psi(x) \end{cases}$$

令 $p(x,t)=\mu(t)+\frac{x}{l}(\nu(t)-\mu(t)) \xRightarrow{\text{易见}} p(0,t)=\mu(t),\ p(l,t)=\nu(t)$

令 $V(x,t)=u(x,t)-p(x,t)$

有
$$\begin{cases} V_{tt}=a^2V_{xx}+f(x,t)-[\mu''(t)+x(\nu''(t)-\mu''(t))/l] \\ V|_{x=0}=V|_{x=l}=0, \\ V|_{t=0}=\varphi(x)-[\mu(0)+x(\nu(0)-\mu(0))/l] \\ V_t|_{t=0}=\psi(x)-[\mu'(0)+x(\nu'(0)-\mu'(0))/l] \end{cases}$$

再按之前所说的方法求解即可

例
$$\begin{cases} u_t=a^2u_{xx} & (0<x<l,\ t>0) \\ u|_{x=0}=u_0,\ u|_{x=l}=u_l \\ u|_{t=0}=\varphi(x) \end{cases}$$

解 令 $v=u-\left[u_0+\frac{x}{l}(u_l-u_0)\right]$

$$\Rightarrow \begin{cases} V_t=a^2V_{xx} & (0<x<l,\ t>0) \\ V|_{x=0}=V|_{x=l}=0 \\ V|_{t=0}=\varphi(x)-\left[u_0+\frac{x}{l}(u_l-u_0)\right] \end{cases}$$

易得 $X(x)=C_n\sin\frac{n\pi}{l}x$

$T(t)=b_ne^{-(\frac{n\pi a}{l})^2t}$

$\therefore V=\sum_{n=1}^{\infty}B_ne^{-(\frac{n\pi a}{l})^2t}\sin\frac{n\pi}{l}x$ 其中 $B_n=\frac{2}{l}\int_0^l\left[u_0+\frac{x}{l}(u_l-u_0)\right]\sin\frac{n\pi}{l}x\,dx$

$\to$ 再解得 u

以下特殊情况，方程和边界条件可同时齐次化

$$\begin{cases} u_{tt}=a^2u_{xx}+f(x) \\ u|_{x=0}=A\text{（常数）}, \quad u|_{x=l}=B\text{（常数）} \\ u|_{t=0}=\varphi(x), \quad u_t|_{t=0}=\psi(x) \end{cases}$$

令 $u(x,t)=V(x,t)+p(x)$，其中 $P(x)$ 满足

$$\begin{cases} a^2p''(x)+f(x)=0 \\ p(0)=A, \quad p(l)=B \end{cases} \xrightarrow[\text{可化为}]{\text{原方程}} \begin{cases} V_{tt}=a^2V_{xx} \quad (0<x<l,\ t>0) \\ V|_{x=0}=0, \quad V_{x=l}=0 \\ V|_{t=0}=\varphi(x)-p(x), \quad V_t|_{t=0}=\psi(x) \end{cases}$$

例：
$$\begin{cases} u_t=a^2u_{xx}+f(x) \quad x\in(0,l), \ t>0 \\ u|_{x=0}=A, \quad u|_{x=l}=B \\ u|_{t=0}=g(x) \end{cases}$$

$$\begin{cases} a^2p''(x)+f(x)=0 \\ p(0)=A, \quad p(l)=B \end{cases}$$

$$p'(x)=-\frac{1}{a^2}\int_0^{\tau} f(t)\,dt$$

$$p(x)=-\frac{1}{a^2}\int_0^x\int_0^{\tau} f(t)\,dt\,d\tau$$

$$\Rightarrow \begin{cases} V_t=a^2V_{xx} \\ V|_{x=0}=0, \quad V_{x=l}=0 \\ V|_{t=0}=g(x)-p(x) \end{cases}$$

$$X(x)=C_n\sin\frac{n\pi}{l}x$$

$$T(t)=e^{-\frac{n^2\pi^2a^2}{l^2}t}$$

$$\therefore V=\sum_{n=1}^{\infty}A_ne^{-\frac{n^2\pi^2a^2}{l^2}t}\sin\frac{n\pi}{l}x$$

$$A_n=\frac{2}{l}\int_0^l\left[g(x)-p(x)\right]\sin\frac{n\pi x}{l}\,dx$$

$$\therefore u=V+p(x)$$

四、周期性条件和自然边界条件

$$\begin{cases} \Delta u=0, & (x,y)\in\Omega: x^2+y^2<1 \\ u|_{\partial\Omega}=\varphi(x,y) & \partial\Omega: x^2+y^2=1 \end{cases}$$

令 $x=r\cos\theta$, $y=r\sin\theta$

$$\begin{cases} u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0, & 0\le r<1,\ 0\le\theta<2\pi \\ u(1,\theta)=\varphi(\theta) \end{cases}$$

令 $u(r,\theta)=R(r)\phi(\theta)$

$$R''(r)\phi(\theta)+\frac{1}{r}R'(r)\phi(\theta)+\frac{1}{r^2}R(r)\phi''(\theta)=0$$

$$\therefore\ -\frac{r^2R''(r)+rR'(r)}{R(r)}=\frac{\phi''(\theta)}{\phi(\theta)}=-\lambda$$

$$(1)\begin{cases} \phi''(\theta)+\lambda\phi(\theta)=0 \\ \phi(\theta)=\phi(\theta+2\pi) \end{cases} \qquad (2)\begin{cases} r^2R''(r)+rR'(r)-\lambda R(r)=0 \\ |R(0)|<+\infty \end{cases}$$

$\lambda>0$，令 $\lambda=\beta^2$

$$\phi(\theta)=C_1\cos\sqrt{\lambda}\theta+C_2\sin\sqrt{\lambda}\theta$$
$$=C_1\cos\beta\theta+C_2\sin\beta\theta$$

由周期性条件 $\beta=n \Rightarrow \lambda_n=n^2\ (n=0,1,2\cdots)$

$$\therefore\ \phi_n(\theta)=C_n\cos n\theta+d_n\sin n\theta$$

把 $\lambda_n=n^2$ 代入 (2) 再由 Eluer 方程（常微分）可得

令 $r=e^t$

$D=\frac{d}{dt}$

$$D(D-1)R+DR-n^2R=0$$
$$\Rightarrow D^2R-n^2R=0$$
$$\Rightarrow \frac{d^2R}{dt^2}-n^2R=0$$

$n \neq 0$ 时 $\therefore R_n = a_n e^{nt} + b_n e^{-nt}$ $n=0$ 时 $R_0 = (C_1 + C_2 t) e^{-0t}$

$= a_n e^{n \ln r} + b_n e^{n \ln \frac{1}{r}}$ $= a_0 + b_0 \ln r$

$= a_n r^n + b_n r^{-n}$

由自然边界条件 $b_0 = b_n = 0$

$\therefore u_0(r,\theta) = R_0(r)\,\Phi_0(r) = a_0 C_0 = \frac{1}{2}\alpha_0$

$u_n(r,\theta) = R_n(r)\,\Phi_n(r) = r^n(\alpha_n \cos n\theta + \beta_n \sin n\theta)$ $(n=1,2,\cdots$

叠加得:

$$u(r,\theta) = \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} r^n(\alpha_n \cos n\theta + \beta_n \sin n\theta)$$

由边值条件 $r=1$

$$\therefore \varphi(\theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}(\alpha_n \cos n\theta + \beta_n \sin n\theta)$$

$\Rightarrow$ $\alpha_n = \frac{1}{\pi}\int_0^{2\pi} \varphi(\theta) \cos n\theta \, d\theta$ $(n=0,1,2\cdots$

$\beta_n = \frac{1}{\pi}\int_0^{2\pi} \varphi(\theta) \sin n\theta \, d\theta$ $(n=1,2\cdots)$

五、幂级数的解法

考虑如下线性二阶线性齐次常微分方程

$$y''+p(x)y'+q(x)y=0$$

定理1. 若 $p(x)=\sum_{n=0}^{\infty}p_n(x-x_0)^n$, $q(x)=\sum_{n=0}^{\infty}q_n(x-x_0)^n$, $|x-x_0|<R$

则方程有解 $y(x)=\sum_{n=0}^{\infty}b_n(x-x_0)^n$, $|x-x_0|<R$ (代入方程比较系数即可)

定理2、若 $(x-x_0)p(x)=\sum_{n=0}^{\infty}p_n(x-x_0)^n$

$(x-x_0)^2q(x)=\sum_{n=0}^{\infty}q_n(x-x_0)^n$, $|x-x_0|<R$

则方程有解：$y(x)=(x-x_0)^c\sum_{n=0}^{\infty}b_n(x-x_0)^n$, 其中 $b_0\neq 0$, b_n, c 待定常数

例：$y''-2xy'+(\lambda-1)y=0$

设 $y=\sum_{n=0}^{+\infty}a_nx^n$, $y'=\sum_{n=1}^{+\infty}na_nx^{n-1}$ $y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-\sum_{n=1}^{\infty}2na_nx^n+\sum_{n=0}^{\infty}a_n(\lambda-1)x^n=0$$

令 $t=n-2$

当 $n=0$ 时 $2na_nx^n=0$
所以可直接加

$$\sum_{t=0}^{\infty}(t+2)(t+1)a_{t+2}x^t-\sum_{n=0}^{\infty}2na_nx^n+\sum_{n=0}^{\infty}a_n(\lambda-1)x^n=0$$

$t=n$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}2na_nx^n+\sum_{n=0}^{\infty}a_n(\lambda-1)x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-2na_n+a_n(\lambda-1)\right]x^n=0$$

$$\therefore\ a_{n+2}=\frac{2n+1-\lambda}{(n+1)(n+2)}a_n,\quad n=0,1,2\cdots$$

$\therefore y = a_0\left[1+\frac{1-\lambda}{2!}x^2+\frac{(1-\lambda)(5-\lambda)}{4!}+\cdots\right]+a_1\left[x+\frac{(3-\lambda)}{3!}x^3+\frac{(3-\lambda)(7-\lambda)}{5!}x^5+\cdots\right]$

当 $\lambda=2n+1$，$n=0,1,2\cdots$ 时，有

$$a_{n+2}=a_{n+4}=\cdots=0$$

当 n 为偶数 即 $n=2m$ 时

$$y_0(x)=a_0\left[1+\frac{(-4)m}{2!}x^2+\frac{(-4)^2m(m-1)}{4!}x^4+\cdots+\frac{(-4)^m m(m-1)(m-2)\cdots 1}{(2m)!}x^{2m}\right]$$

当 n 为奇数时，即 $n=2m+1$ 时

$$y_1(x)=a_1\left[x+\frac{-4m}{3!}x^3+\frac{(-4)^2m(m-1)}{5!}x^5+\cdots+\frac{(-4)^m m(m-1)\cdots 1}{(2m+1)!}x^{2m+1}\right]$$

积分变换法

一、傅里叶变换及其性质

若 $f(x)$ 在 $(-\infty,+\infty)$ 上绝对可积，称

$$g(\lambda)=\int_{-\infty}^{+\infty} f(x)e^{-i\lambda x}dx = F[f],\ \lambda\in(-\infty,+\infty)$$

为 $f(x)$ 的傅里叶变换

称 $f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} g(\lambda)e^{i\lambda x}d\lambda = F^{-1}[g]$ 为 $g(\lambda)$ 的傅里叶逆变换

$$F[e^{-x^2}]=\sqrt{\pi}e^{-\frac{\lambda^2}{4}} \qquad F[e^{-Ax^2}]=\sqrt{\frac{\pi}{A}}\,e^{-\frac{\lambda^2}{4A}}$$

$$F\left[\frac{1}{a^2+x^2}\right]=\int_{-\infty}^{+\infty}\frac{1}{a^2+x^2}\cdot e^{-i\lambda x}dx \quad (a>0)$$

当 $\lambda\le 0$ 时 $F\left[\frac{1}{a^2+x^2}\right]=2\pi i\,\mathrm{Res}\left[\frac{e^{-i\lambda z}}{a^2+z^2};ai\right]$

$$=2\pi i\cdot\frac{e^{-i\lambda z}}{2z}\Big|_{z=ai}=\frac{\pi}{a}e^{a\lambda}$$

当 $\lambda>0$ 时 $F\left[\frac{1}{a^2+x^2}\right]\overset{x=-t}{=\!=\!=}\int_{-\infty}^{+\infty}\frac{1}{a^2+t^2}e^{i\lambda t}dt=\frac{\pi}{a}e^{-a\lambda}$

基本性质：

(1) 线性性质

$F[\alpha_1 f_1+\alpha_2 f_2]=\alpha_1F[f_1]+\alpha_2F[f_2]$，$\alpha_1,\alpha_2\in C$

(2) 微分性质

若 $\lim\limits_{x\to\infty}f(x)=0$，则 $F[f'(x)]=i\lambda F[f(x)]$

$\Rightarrow \lim\limits_{x\to\infty}f^{(i)}(x)=0,\ i=0,1,2,\cdots,m-1$

$\Rightarrow F[f^{(m)}(x)]=(i\lambda)^m F[f(x)]$

卷积、 $f_1 * f_2(x) = \int_{-\infty}^{+\infty} f_1(x-t) f_2(t) dt$

(3) 卷积性质

$$F[f_1 * f_2] = F[f_1] F[f_2]$$

$$F[f_1 \cdot f_2(x)] = \frac{1}{2\pi} F[f_1] * F[f_2]$$

(4) 平移性质

$$F[f(x-a)] = e^{-i\lambda a} F[f] \quad a \in R$$

$$F[e^{iax} f(x)] = F[f](\lambda - a)$$

(5) 伸缩性质

$$F[f(kx)] = \frac{1}{|k|} F[f](\frac{\lambda}{k}), \ k \neq 0, k \in R$$

(6) 乘子性质

$$F[x f(x)] = i(F[f])'$$

$$F[x^m f(x)] = i^m (F[f(x)])^{(m)}$$

(7) 对称性

$$F^{-1}[f(x)](\lambda) = \frac{1}{2\pi} F[f(x)](-\lambda)$$

或 $F[f(x)](\lambda) = 2\pi F^{-1}[f(x)](-\lambda)$

δ函数的性质：

$$\int_{-\infty}^{+\infty} \delta(x) dx = 1$$

对 $\forall f(x) \in C(-\infty, +\infty)$

$\text{令 } \varepsilon \to 0 \quad \int_{-\infty}^{+\infty} \delta(x) f(x) dx = f(0)$

一、一维热传导方程的初值问题

$$\begin{cases} u_t - a^2 u_{xx} = f(x,t) & (-\infty < x + \infty, t > 0) \\ u|_{t=0} = \varphi(x), & (-\infty < x + \infty) \end{cases}$$

对x作傅里叶变换：记

$$F(u) = \hat{u}(\lambda, t), \quad F[f] = \hat{f}(\lambda, t) \quad F[\varphi(x)] = \hat{\varphi}(\lambda)$$

两边同时做傅里叶变换，利用微分性质

$$\therefore \begin{cases} \frac{d\hat{u}}{dt} - (i\lambda)^2 a^2 \hat{u} = \hat{f}(\lambda, t) \\ \hat{u}|_{t=0} = \hat{\varphi}(\lambda) \end{cases} \Rightarrow \begin{cases} \frac{d\hat{u}}{dt} + \lambda^2 a^2 \hat{u} = \hat{f}(\lambda, t) \\ \hat{u}|_{t=0} = \hat{\varphi}(\lambda) \end{cases}$$

一阶线性常微分方程公式：

$$\hat{u} = e^{-\int_0^t \lambda^2 a^2 d\xi} \left(\int_0^t \hat{f}(\lambda, \tau) e^{\int_0^\tau \lambda^2 a^2 d\xi} d\tau + C \right)$$

$$\because \hat{u}|_{t=0} = \hat{\varphi}(\lambda) \quad \therefore C = \hat{\varphi}(\lambda)$$

$$\therefore \hat{u} = e^{-\lambda^2 a^2 t} \hat{\varphi}(\lambda) + \int_0^t \hat{f}(\lambda, \tau) e^{-\lambda^2 a^2 (t-\tau)} d\tau$$

两边同时取 F^{-1} 得，(卷积性质) $F[f_1 * f_2] = F[f_1] \cdot F[f_2]$

$$u(x,t) = F^{-1}[e^{-\lambda^2 a^2 t}] * \varphi(x) + \int_0^t f(x,\tau) * F^{-1}[e^{-\lambda^2 a^2 (t-\tau)}] d\tau$$

由前结论可得 $F[e^{-Ax^2}] = \sqrt{\frac{\pi}{A}} e^{-\frac{\lambda^2}{4A}}$

取 $4A = \frac{1}{a^2 t}$

$$\therefore \sqrt{\frac{\pi}{\frac{1}{4a^2 t}}} e^{-\lambda^2 a^2 t} = F[e^{-\frac{x^2}{4a^2 t}}]$$

$$\therefore e^{-\lambda^2 a^2 t} = \frac{1}{2a\sqrt{\pi t}} F[e^{-\frac{x^2}{4a^2 t}}]$$

$$\therefore F^{-1}[e^{-\lambda^2 a^2 t}] = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}}$$

$$\therefore u(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi+\int_0^t\frac{1}{2a\sqrt{\pi(t-\tau)}}\int_{-\infty}^{+\infty}f(\xi,\tau)e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}d\xi d\tau$$

二、椭圆型边值问题

记 $R_+^2=\{(x,y):-\infty<x<+\infty,\ y>0\}$

$$\begin{cases}u_{xx}+u_{yy}=0, & (x,y)\in R_+^2\\ u(x,0)=f(x), & -\infty<x<+\infty\\ u\text{有界}\end{cases}$$

对 x 作傅里叶变换 记 $F(u)=\hat{u}$，$F[f(x)]=\hat{f}(x)$

$$\begin{cases}\frac{d^2\hat{u}}{dy^2}-\lambda^2\hat{u}=0\\ \hat{u}(\lambda,0)=\hat{f}(\lambda)\\ \hat{u}\text{关于}y\text{有界}\end{cases}$$

$\hat{u}=c_1e^{|\lambda|y}+c_2e^{-|\lambda|y}$

代入 $y=0$ 得 $\hat{u}=C_1+C_2=\hat{f}(\lambda)$

$\because \hat{u}$ 关于 y 有界 $\therefore C_1=0$，$C_2=\hat{f}(\lambda)$

$\therefore \hat{u}=\hat{f}(\lambda)e^{-|\lambda|y}$

$$\therefore u(x,y)=\frac{1}{\pi}\int_{-\infty}^{+\infty}f(\xi)\frac{y}{(x-\xi)^2+y^2}d\xi$$

两边取 F^{-1}

$u(x,y)=f(x)*F^{-1}[e^{-|\lambda|y}]$ 进一步可以

$\min\limits_{-\infty<x<+\infty}f(x)\le u(x,y)\le\max\limits_{-\infty<x<+\infty}f(x)$

$$F^{-1}[e^{-|\lambda|y}]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-|\lambda|y}e^{i\lambda x}d\lambda$$

$$=\frac{1}{2\pi}\int_{-\infty}^{0}e^{\lambda y+i\lambda x}d\lambda+\frac{1}{2\pi}\int_0^{+\infty}e^{-\lambda y+i\lambda x}d\lambda$$

$$=\frac{1}{2\pi}\left(\frac{1}{y+ix}+\frac{1}{y-ix}\right)=\frac{1}{\pi}\frac{y}{x^2+y^2}$$